Indian Statistical Institute
M. Math 2nd year
Academic year 2022-2023
Backpaper Examination
Course: Special Topics in Geometry: Harmonic maps
07 - 06 - 2023
3 hours

- *Answer as many questions as you can.*
- *You may use results proved in class, but make sure to state them clearly.*
- *Maximum marks is 100.*

1. Let $M_n(\mathbb{R})$ denote the vector space of $n \times n$ real matrices equipped with the operator norm.

   Let $f : M_n(\mathbb{R}) \to \mathbb{R}$ be defined by $f(A) := \det A$ for $A \in M_n(\mathbb{R})$.

   Show that f is differentiable and compute the derivative $Df_A : M_n(\mathbb{R}) \to \mathbb{R}$ for any invertible $n \times n$ matrix A.

   (12 marks)

2. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Suppose that $D^2F_x(v,v) > 0$ for all $v \in \mathbb{R}^n - \{0\}$ and for all $x \in \mathbb{R}^n$. Suppose that $F(x) \to +\infty$ as $||x|| \to \infty$. Show that F has a unique minimum x_0 in $\mathbb{R}^n$. (14 marks)

3. Let M be a smooth manifold, let X be a complete vector field on M and let $(\phi_t : M \to M)_{t\in\mathbb{R}}$ be the flow of X. Let ω be a closed k-form on M, i.e. $d\omega = 0$. Let $\sigma = \sum_{i=1}^m a_i c_i$ be a singular k-chain on M (where $m \geq 1$, $a_i \in \mathbb{R}$, and c_i is a singular k-cube in M, for $i = 1, \ldots, m$). Suppose that σ is a cycle, i.e. $\partial\sigma = 0$. Show that there is a constant $k \in \mathbb{R}$ such that

$$\sum_{i=1}^m a_i \cdot \int_{\phi_t \circ c_i} \omega = k$$

   for all $t \in \mathbb{R}$. (12 marks)

4. Let M be a compact Riemannian manifold without boundary. Let f be a smooth function on M such that all integral curves of the gradient vector field ∇f are geodesics. Show that f is constant. (12 marks)

5. Let M be a Riemannian manifold. Let $p \in M$ and let $y_1, \ldots, y_n$ be normal coordinates near p. Let f be a smooth function on M.

   (a) Show that $\nabla f(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(p) \frac{\partial}{\partial y_i}_p$.

   (b) Show that $\Delta f(p) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial y_i^2}(p)$.

   (6+8 = 14 marks)

6. Let M, N be Riemannian manifolds. Let $f : M \to N$ be a smooth map. Show that f is totally geodesic if and only if, for any $p \in M$, if u is a smooth convex function in a neighbourhood of $f(p)$ in N, then $u \circ f$ is convex in a neighbourhood of p. (18 marks)

7. Let M be a compact Riemannian manifold without boundary, and suppose the sectional curvature of M is nonpositive. Show that the higher homotopy groups $\pi_n(M)$ are zero for all $n \geq 2$. (10 marks)

8. Let M be a compact Riemannian manifold without boundary, and suppose the sectional curvature of M is strictly negative everywhere. Let $p \in M$, and let $a, b : [0, 1] \to M$ be loops based at p. Suppose that the corresponding elements of $\pi_1(M, p)$ commute, $[a] \cdot [b] = [b] \cdot [a]$. Show that there is an element $[c] \in \pi_1(M, p)$ and integers $m, n \in \mathbb{Z}$ such that $[a] = [c]^m, [b] = [c]^n$. (18 marks)